# MIXED TÊTE-À-TÊTE TWISTS AS MONODROMIES ASSOCIATED WITH HOLOMORPHIC FUNCTION GERMS

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Abstract. Tête-à-tête graphs were introduced by N. A'Campo in 2010 with the goal of modeling the monodromy of isolated plane curves. Mixed tête-à-tête graphs provide a generalization which define mixed tête-à-tête twists, which are pseudo-periodic automorphisms on surfaces. We characterize the mixed tête-à-tête twists as those pseudo-periodic automorphisms that have a power which is a product of right-handed Dehn twists around disjoint simple closed curves, including all boundary components. It follows that the class of tête-à-tête twists coincides with that of monodromies associated with reduced function germs on isolated complex surface singularities.

Contents



## 1. Introduction

In [A'C] N. A'Campo introduced the notion of pure tête-à-tête graph in order to model monodromies of plane curves. These are metric ribbon graphs without univalent vertices that satisfy a special property. If one sees the ribbon graph $\Gamma$ as a strong deformation retract of a surface $\Sigma$ with boundary, the tête-à-tête propety says that starting at any point $p$ and walking a distance of $\pi$ in any direction and always turning right at vertices, gets you to the same point. This property defines an element in the mapping class group $\mathrm{MCG}^+(\Sigma, \partial\Sigma)$ which is freely periodic, and is called the *tête-à-tête twist* associated with $\Gamma$.

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In [Gra14], C. Graf proved that if one allows univalent vertices in tête-à-tête graphs, then one is able to model all freely periodic mapping classes of $\mathrm{MCG}^+(\Sigma, \partial\Sigma)$ with positive fractional Dehn twist coefficients. In [FPP] this result was improved by showing that one does not need to enlarge the original class of metric ribbon graphs used to prove the same theorem.

However, the geometric monodromy of an isolated plane curve singularity with one branch and more than one Puiseux pair is not of finite order (see [A'C73]). For this purpose, the definition of mixed tête-à-tête graph was introduced in [FPP]. These are metric ribbon graphs together with a filtration and a set of locally constant functions that model a class of pseudo-periodic automorphisms. It was proved that this class includes the monodromy associated with an isolated singularity on a plane curve with one branch.

In this work we continue the study of (relative) mixed tête-à-tête graphs and we improve and generalize results from [FPP]. The main result is a complete characterization of the mapping classes that can be modeled by a mixed tête-à-tête graph. This is the content of Theorem 6.7 which says

**Theorem A.** *Let $\phi : \Sigma \to \Sigma$ be an automorphism with fixes the boundary. Then there exists a mixed tête-à-tête graph in $\Sigma$ inducing its mapping class in $\mathrm{MCG}^+(\Sigma, \partial\Sigma)$ if and only if some power of $\phi$ is a composition of right handed Dehn twists around disjoint simple closed curves including all boundary components.*

A reduced holomorphic function germ on an isolated surface singularity has an associated Milnor fibration with a monodromy which fixes the boundary. It is known that this monodromy is pseudo-periodic and a power of it is a composition of right-handed Dehn twists around disjoint simple closed curves, which include all the boundary curves. It follows that the Milnor fiber associated with such a function germ contains a mixed tête-à-tête graph which defines the monodromy. Conversely, a result by Neumann and Pichon [NP07] says that any such a surface automorphism is realized as the monodromy associated with a function germ. Hence

**Theorem B.** *Mixed tête-à-tête twists are precisely the monodromies associated with reduced function germs on isolated surface singularities.*

The structure of the work is the following. In Section 2 we briefly introduce notation related to the mapping class group that we use throughout the text.

In Section 3 we fix notation and conventions about pseudo-periodic automorphisms of surfaces. Not all of this section is contained in [FPP] since we treat a broader class of pseudo-periodic automorphisms in the present text. In particular we allow amphidrome orbits of annuli. This section is important for Section 6 which contains the main results.

In Sections 4 and 5 we recall some necessary definitions and results about pure and mixed tête-à-tête graphs.

Section 6 starts with the statement of the main Theorem 6.7. Two Lemmas which are used in the proof follow and the section ends with the proof of the main result.

The work ends with an example that depicts a mixed tête-à-tête graph.

## 2. The mapping class group

**2.1.** Let $\Sigma$ be a surface with $\partial\Sigma \neq \emptyset$. We denote by $\mathrm{MCG}(\Sigma)$ the mapping class group given by the automorphisms of $\Sigma$ up to isotopy, where the automorphisms of the isotopy do not neccesarily fix the boundary. Let $\partial^1\Sigma \subset \partial\Sigma$ be a subset formed by some boundary components of $\Sigma$. We will denote by $\mathrm{MCG}(\Sigma, \partial^1\Sigma)$ the mapping class group given by automorphisms of $\Sigma$ that are the identity restricted to $\partial^1\Sigma$ and where isotopies are along automorphisms that are the identity on these boundary components.

Let $\phi : \Sigma \to \Sigma$ be a automorphism, we denote its class in $\mathrm{MCG}(\Sigma)$ by $[\phi]$. If $\phi|_{\partial^1\Sigma} = \mathrm{id}$ we denote its class in $\mathrm{MCG}(\Sigma, \partial^1\Sigma)$ by $[\phi]_{\partial^1\Sigma}$.

Given two automorphisms $\phi$ and $\psi$ of $\Sigma$ that both leave invariant some subset $B \subset \partial\Sigma$ such that $\phi|_B = \psi|_B$, we say they are *isotopic relative to the action* $\phi|_B$ if there exists a family of automorphisms of $\Sigma$ that isotope them as before and such that any automorphism of the family has the same restriction to $B$ as $\phi$ and $\psi$. We write $[\phi]_{B,\phi|_B} = [\psi]_{B,\phi|_B}$. We denote by $\mathrm{MCG}(\Sigma, B, \phi|_B)$ the set of classes $[\phi]_{B,\phi|_B}$ with respect to this equivalence relation. We denote by $\mathrm{MCG}^+(\Sigma, B, \phi|_B)$ if we restrict to automorphisms preserving orientation.

**2.2.** Consider an automorphism $\phi : \Sigma \to \Sigma$ with $\phi|_{\partial^1\Sigma} = \mathrm{id}$ for some subset $\partial^1\Sigma \subset \partial\Sigma$. Let $\mathcal{D}_i$ denote a right-handed Dehn twist around a curve parallel to the boundary component $C_i \subset \partial^1\Sigma$. Suppose that $[\phi] \in \mathrm{MCG}(\Sigma)$ is of finite order, i.e. there exists a natural number $n$ such that $[\phi]^n = [\mathrm{id}]$. Then we have that $[\phi^n]_{\partial^1\Sigma} = [\mathcal{D}_1]^{t_1}_{\partial^1\Sigma} \cdots [\mathcal{D}_r]^{t_r}_{\partial^1\Sigma}$ with $t_i \in \mathbb{Z}$. We call $t_i/n$ the *fractional Dehn twist coefficient* of $\phi$ at the component $C_i$.

## 3. Pseudo-periodic automorphisms

We recall conventions, definitions and results from [FPP] Part II. that we will use in the present work. We also extend some of the notions there to cover some cases that were not treated in that work.

**Definition 3.1.** A automorphism $\phi : \Sigma \to \Sigma$ is pseudo-periodic if it is isotopic to a automorphism satisfying that there exists a finite collection of disjoint simple closed curves $\mathcal{C}$ such that

(i) $\phi(\mathcal{C}) = \mathcal{C}$.
(ii) $\phi|_{\Sigma\setminus\mathcal{C}}$ is freely isotopic to a periodic automorphism.

Assuming that none of the connected components of $\Sigma \setminus \mathcal{C}$ is either a disk or an annulus and that the set of curves is minimal, which is always possible, we name $\mathcal{C}$ an admissible set of curves for $\phi$.

The following theorem is a particularization on pseudo-periodic automorphisms of the more general Corollary 13.3 in [FM12] that describes a *canonical form* for every automorphism of a surface.

**Theorem 3.2** (Almost-Canonical Form and Canonical Form)**.** *Let $\Sigma$ be a surface with $\partial\Sigma \neq \emptyset$. Any pseudo-periodic map of $\Sigma$ is isotopic to an automorphism in* almost-canonical form*, that means a automorphism $\phi$ which has an admissible set of curves $\mathcal{C} = \{\mathcal{C}_i\}$ and annular neighborhoods $\mathcal{A} = \{\mathcal{A}_i\}$ with $\mathcal{C}_i \subset \mathcal{A}_i$ such that*

*(i) $\phi(\mathcal{A}) = \mathcal{A}$.*
*(ii) The map $\phi|_{\overline{\Sigma\setminus\mathcal{A}}}$ is periodic.*

*When the set $\mathcal{C}$ is minimal we say that $\phi$ is in canonical form.*

**Remark 3.3.** In the case we have a pseudo-periodic automorphism of $\Sigma$ that fixes pointwise some components $\partial^1\Sigma$ of the boundary $\partial\Sigma$ we can always find *a* *canonical form* as follows. We can find an isotopic automorphism $\phi$ relative to $\partial^1\Sigma$ that coincides with a canonical form as in the previous theorem outside a collar neighborhood $U$ of $\partial^1\Sigma$. We may assume that there exists an isotopy connecting the automorphism and its canonical form relative to $\partial^1\Sigma$.

**3.4.** Let $\phi$ be a pseudo-periodic automorphism in some almost-canonical form. Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be a subset of curves in $\mathcal{C}$ that are cyclically permuted by $\phi$, i.e. $\phi(\mathcal{C}_i) = \mathcal{C}_{i+1}$ for $i = 1, \ldots, k-1$ and $\phi(\mathcal{C}_k) = C_1$. Suppose that we give an orientation to $\mathcal{C}_1, \ldots, \mathcal{C}_k$ so that $\phi|_{\mathcal{C}_i}$ for $i = 1, \ldots, k-1$ is orientation preserving. We say that the curves are *amphidrome* if $\phi|_{\mathcal{C}_k} : \mathcal{C}_k \to \mathcal{C}_1$ is orientation reversing.

**Notation 3.5.** Let $s, c \in \mathbb{R}$. We denote by $\mathcal{D}_{s,c}$ the automorphism of $\mathbb{S}^1 \times I$ given by by $(x,t) \mapsto (x + st + c, t)$ (we are taking $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$). Observe that

$$\mathcal{D}_{s,c} \circ \mathcal{D}_{s',c'} = \mathcal{D}_{s+s',c+c'},$$
$$\mathcal{D}_{s,c}^{-1} = \mathcal{D}_{-s,-c}.$$

In this work we always have $s \in \mathbb{Q}$.

**Remark 3.6.** We can isotope $\mathcal{D}_{s,c}$ to a automorphism $\mathcal{D}^p_{s,c}$ that is periodic on some tubular neighborhood of the core curve $\mathbb{S}^1 \times \{1/2\}$ of the annulus while preserving the action on the boundary $\partial(\mathbb{S}^1 \times I)$:

$$\mathcal{D}^p_{s,c}(x,t) = \begin{cases} (x + 3s(t - \frac{1}{3}) + c, t) & 0 \leq t \leq \frac{1}{3} \\ (x + c, t) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ (x + 3s(t - \frac{2}{3}) + c, t) & \frac{2}{3} \leq t \leq 1 \end{cases} \tag{3.7}$$

**Notation 3.8.** We denote by $\tilde{\mathcal{D}}_s$ the automorphism of $\mathbb{S}^1 \times I$ given by

$$\tilde{\mathcal{D}}_s(x,t) = \begin{cases} (-x - 3s(t - \frac{1}{3}), 1 - t) & 0 \leq t \leq \frac{1}{3} \\ (-x, 1 - t) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ (-x - 3s(t - \frac{2}{3}), 1 - t) & \frac{2}{3} \leq t \leq 1 \end{cases} \tag{3.9}$$

In this case we only work with $s \in \mathbb{Q}$ as well.

**Definition 3.10.** Let $\mathcal{C} \subset \Sigma$ be a simple closed curve embedded in an oriented surface $\Sigma$. And let $\mathcal{A}$ be a tubular neighborhood of $\mathcal{C}$. Let $\mathcal{D} : \Sigma \to \Sigma$ be a automorphism of the surface with $\mathcal{D}|_{\Sigma \setminus \mathcal{A}} = \mathrm{id}$. We say that $\mathcal{D}$ is a negative Dehn twist around $\mathcal{C}$ or a right-handed Dehn twist if there exist a parametrization $\eta : \mathbb{S}^1 \times I \to \mathcal{A}$ preserving orientation such that

$$\mathcal{D} = \eta \circ \mathcal{D}_{1,0} \circ \eta^{-1}.$$

A positive Dehn twist is defined similarly changing $\mathcal{D}_{1,0}$ by $\mathcal{D}_{-1,0}$ in the formula above.

**Lemma 3.11** (Linearization. Equivalent to Lemma 2.1 in [MM11])**.** *Let $\mathcal{A}_i$ be an annulus and let $\phi : \mathcal{A}_i \to \mathcal{A}_i$ be a automorphism that does not exchange boundary components. Suppose that $\phi|_{\partial\mathcal{A}_i}$ is periodic. Then, after an isotopy fixing the boundary, there exists a parametrization $\eta : \mathbb{S}^1 \times I \to \mathcal{A}_i$ such that*

$$\phi = \eta \circ \mathcal{D}_{-s,-c} \circ \eta^{-1}$$

*for some $s \in \mathbb{Q}$, some $c \in \mathbb{R}$.*

**Lemma 3.12** (Specialization. Equivalent to Lemma 2.3 in [MM11])**.** *Let $\mathcal{A}_i$ be an annulus and let $\phi : \mathcal{A}_i \to \mathcal{A}_i$ be a automorphism that exchanges boundary components. Suppose that $\phi|_{\partial \mathcal{A}_i}$ is periodic. Then after an isotopy fixing the boundary, there exists a parametrization $\eta : \mathbb{S}^1 \times I \to \mathcal{A}_i$ such that*

$$\phi = \eta \circ \tilde{\mathcal{D}}_{-s} \circ \eta^{-1}$$

*for some $s \in \mathbb{Q}$.*

**Remark 3.13.** In the case $\phi|_{\partial \mathcal{A}_i}$ is the identity, we have that

$$\phi = \eta \circ \mathcal{D}_{s,0} \circ \eta^{-1}$$

for some $s \in \mathbb{Z}$, that is $\phi = \mathcal{D}^s$ for some right-handed Dehn twist as in Definition 3.10.

**Definition 3.14** (Screw number)**.** Let $\phi$ be a pseudo-periodic automorphism as in Theorem 3.2. Let $n$ be the order of $\phi|_{\Sigma \setminus \mathcal{A}}$. By Remark 3.13, $\phi^n|_{\mathcal{A}_i}$ equals $\mathcal{D}|_{\mathcal{A}_i}^{s_i}$ for a certain $s_i \in \mathbb{Z}$.

Let $\alpha$ be the length of the orbit in which $\mathcal{A}_i$ lies and let $\tilde{\alpha} \in \{\alpha, 2\alpha\}$ be the smallest number such that $\phi^{\tilde{\alpha}}$ does not exchange the boundary components of $\mathcal{A}_1$. We define

$$s(\mathcal{A}_i) := \frac{-s_i}{n}\tilde{\alpha}.$$

We call $s(\mathcal{A}_i)$ the *screw number* of $\phi$ at $\mathcal{A}_i$ or at $\mathcal{C}_i$.

**Remark 3.15.** Compare Definition 3.14 with [[MM11] p.4] and with Definition 2.4 in [MM11]. The original definition is due to Nielsen [[Nie44]. Sect. 12] and it does not depend on a canonical form for $\phi$. Since we are restricting to automorphisms that do not exchange boundary components of the annuli $\mathcal{A}$, our definition is a bit simpler.

**Lemma 3.16.** *Let $\phi$ be a automorphism as in Theorem 3.2 and let $\{\mathcal{A}_i\} \subset \mathcal{A}$ be a set of $k$ annuli cyclically permuted by $\phi$, i.e. $\phi(\mathcal{A}_i) = \mathcal{A}_{i+1}$ such that $\phi^k$ does not exchange boundary components. Then there exist coordinates*

$$\eta_i : \mathbb{S}^1 \times I \to \mathcal{A}_i$$

*for the annuli in the orbit such that*

$$\eta_{j+1}^{-1} \circ \phi \circ \eta_j = \mathcal{D}_{-s/k, -c/k}$$

*where $s$ and $c$ are associated to $\mathcal{A}_1$ as in Lemma 3.11.*

*Proof.* See [FPP] □

**Remark 3.17.** By Remark 3.6 we can substitute $\mathcal{D}_{-s/k,-c/k}$ by $\mathcal{D}^p_{-s/k,-c/k}$ in the previous lemma.

**Lemma 3.18.** *Let $\phi$ be a automorphism as in Theorem 3.2 and let $\{\mathcal{A}_i\} \subset \mathcal{A}$ be a set of $k$ annuli cyclically permuted by $\phi$, i.e. $\phi(\mathcal{A}_i) = \mathcal{A}_{i+1}$ such that $\phi^k$ exchanges boundary components. Then there exist coordinates*

$$\eta_i : \mathbb{S}^1 \times I \to \mathcal{A}_i$$

*for the annuli in the orbit such that*

$$\eta_{j+1}^{-1} \circ \phi \circ \eta_j = \tilde{\mathcal{D}}_{-s/\alpha}$$

*where $s$ is associated to $\mathcal{A}_1$ as in Lemma 3.12.*

*Proof.* Take a parametrization of $\mathcal{A}_1$ for $\phi^k : \mathcal{A}_1 \to \mathcal{A}_1$ as in Lemma 3.12, say $\eta_1 : \mathbb{S}^1 \times I \to \mathcal{A}_1$. Define recursively $\eta_j := \phi \circ \eta_{j-1} \circ \tilde{\mathcal{D}}_{s/k}$ (see Notation 3.5). Then, we have

$$\eta_{j+1}^{-1} \circ \phi \circ \eta_j = \tilde{\mathcal{D}}_{-s/k}.$$

Since for every $j$ we have that $\eta_j = \phi^{j-1} \circ \eta_1 \circ \tilde{\mathcal{D}}_{s(j-1)/k}$ we have also that

$$\eta_1^{-1} \circ \phi \circ \eta_\alpha = \eta_1^{-1} \circ \phi \circ \phi^{k-1} \circ \eta_1 \circ \tilde{\mathcal{D}}_{s(k-1)/k} = \tilde{\mathcal{D}}_{-s/k}.$$

□

**Remark 3.19.** After this proof we can check that $\eta_k^{-1} \circ \phi^\alpha \circ \eta_k = \mathcal{D}_{-s,-c}$ to see that the screw number $s = s(\mathcal{A}_i)$ and the parameter $c$ modulo $\mathbb{Z}$ of Lemma 3.16 only depend on the orbit of $\mathcal{A}_i$.

We observe also that the numbers $s$ and $c$ of Lemma 3.16 satisfy

- ❀ $s$ equals $s(\mathcal{A}_i)$ and
- ❀ $c$ is only determined modulo $\mathbb{Z}$ and equals the usual rotation number $\mathrm{rot}(\phi^{\alpha_i}|_{\eta(\mathbb{S}^1 \times \{0\})}) \in (0, 1]$.

This is also observed in Corollary 2.2 in [MM11].

**Definition 3.20.** Let $C$ be a component of $\partial\Sigma$ and let $\mathcal{A}$ be a compact collar neighborhood of $C$ in $\Sigma$. Suppose that $C$ has a metric and total length is equal to $\ell$. Let $\eta : \mathbb{S}^1 \times I \to \mathcal{A}$ be a parametrization of $\mathcal{A}$, such that $\eta|_{\mathbb{S}^1 \times \{1\}} : \mathbb{S}^1 \times \{1\} \to C$ is an isometry.

Suppose that $\mathbb{S}^1$ has the metric induced from taking $\mathbb{S}^1 = \mathbb{R}/\ell\mathbb{Z}$ with $\ell \in \mathbb{R}_{>0}$ and the standard metric on $\mathbb{R}$. A *boundary Dehn twist* of length $r \in \mathbb{R}_{>0}$ along $C$ is a automorphism $\mathcal{D}_r^\eta(C)$ of $\Sigma$ such that:

(i) it is the identity outside $\mathcal{A}$

(ii) the restriction of $\mathcal{D}_r^\eta(C)$ to $\mathcal{A}$ in the coordinates given by $\eta$ is given by $(x, t) \mapsto (x + r \cdot t, t)$.

The isotopy type of $\mathcal{D}_r^\eta(C)$ by isotopies fixing the action on $\partial\Sigma$ does not depend on the parametrization $\eta$. When we write just $\mathcal{D}_r(C)$, it means that we are considering a boundary Dehn twist with respect to *some* parametrization $\eta$.

**Remark 3.21.** Given a automorphism $\phi$ of a surface $\Sigma$ with $\partial\Sigma \neq \emptyset$. Let $C$ be a connected component of $\partial\Sigma$ such that $\phi|_C$ is a rotation by $c \in [0, 1)$. Let $\mathcal{A}$ be a compact collar neighborhood of $C$ (isomorphic to $I \times C$) in $\Sigma$. Let $\eta : \mathbb{S}^1 \times I \to \mathcal{A}$ be a parametrization of $\mathcal{A}$, with $\phi(\mathbb{S}^1 \times \{1\}) = C$. Up to isotopy, we can assume that the restriction of $\phi$ to $\mathcal{A}$ satisfies

$$\eta^{-1} \circ \phi|_{\mathcal{A}} \circ \eta(x, t) = (x + c, t).$$

## 4. Pure tête-à-tête graphs

In this section we recall some definitions and conventions from [FPP] Part I.

**4.1.** A graph $\Gamma$ is a 1 dimensional finite CW-complex; unless otherwise specified a graph doesn't have univalent vertices. A ribbon graph is a graph equipped with a cyclic ordering of the edges adjacent to each vertex. With a ribbon graph, one can recover the topology of an orientable surface with boundary, we call this surface the *thickening* of $\Gamma$. A metric ribbon graph is a ribbon graph with a metric on each of its edges.

A relative metric ribbon graph is a pair $(\Gamma, A)$ with $A \subset \Gamma$ a subgraph formed by a disjoint union of circles with the property that for each connected component $A_i \subset A$, there exists a boundary component on the thickening $\Sigma$ of $\Gamma$ such that it retracts to $A_i$. The relative thickening of $(\Gamma, A)$ is the thickening of $\Gamma$ minus the cylinders corresponding to the boundary components that retract to $A$. In particular, the relative thickening, also denoted by $\Sigma$ contains $A$ as boundary.

**Definition 4.2** (Safe walk)**.** Let $(\Gamma, A)$ be a metric relative ribbon graph. A safe walk for a point $p$ in the interior of some edge is a path $\gamma_p : \mathbb{R}_{\geq 0} \to \Gamma$ with $\gamma_p(0) = p$ and such that:

(1) The absolute value of the speed $|\gamma_p'|$ measured with the metric of $\Gamma$ is constant and equal to 1. Equivalently, the safe walk is parametrized by arc length, i.e. for $s$ small enough $d(p, \gamma_p(s)) = s$.
(2) when $\gamma_p$ gets to a vertex, it continues along the next edge in the given cyclic order.
(3) If $p$ is in an edge of $A$, the walk $\gamma_p$ starts running in the opposite direction to the one indicated by $A$ seen as boundary of $\Sigma$.

An $\ell$-safe walk is the restriction of a safe walk to the interval $[0, \ell]$. If a length is not specified when referring to a safe walk, we will understand that its length is $\pi$.

The notion in (2) of *continuing along the next edge in the order* of $e(v)$ is equivalent to the notion of *turning to the right* in every vertex for paths parallel to $\Gamma$ in $\Sigma$ in A'Campo's words in [A'C].

**Definition 4.3** (Tête-à-tête property)**.** Let $(\Gamma, A)$ be relative metric ribbon graph. We say that $\Gamma$ satisfies the *$\ell$-tête-à-tête property*, or that $\Gamma$ is an *$\ell$-tête-à-tête graph* if

1) For any point $p \in \Gamma \setminus (A \cup v(\Gamma))$ the two different $\ell$-safe walks starting at $p$, that we denote by $\gamma_p, \omega_p$, satisfy $\gamma_p(\ell) = \omega_p(\ell)$.
2) for a point $p$ in $A \setminus v(\Gamma)$, the end point of the unique $\ell$-safe walk starting at $p$ belongs to $A$.

In this case, we say that $(\Gamma, A)$ is a *relative $\ell$-tête-à-tête graph.* If $A = \emptyset$, we call it a *pure* $\ell$-tête-à-tête structure or graph. If $\ell = \pi$ we just call it pure tête-à-tête structure or graph.

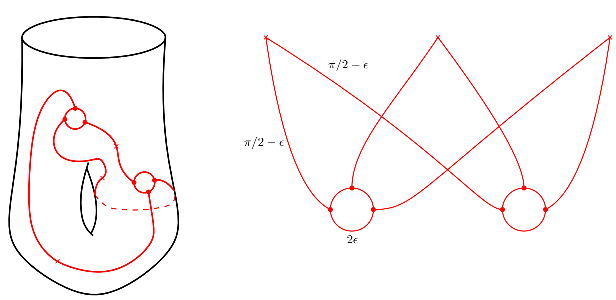


**Figure 4.1.** An example of a relative tête-à-tête graph. It has 2 connected components in $A$ (the two small circles). The length of an edge in $A$ is $2\varepsilon$ and the length of and edge from a vertex in $A$ to a vertex depicted as a cross is $\pi/2 - \varepsilon$.

**Notation 4.4.** Let $(\Gamma, A) \hookrightarrow (\Sigma, \partial\Sigma)$ be a relative ribbon graph properly embedded in its thickening. Let $\Sigma_\Gamma$ be the surface that results from cutting $\Sigma$ along $\Gamma$, then $\Sigma_\Gamma$ consists of as many cylinders as there are connected components in $\partial\Sigma \setminus A$. We denote these cylinders by $\tilde{\Sigma}_1, \dots, \tilde{\Sigma}_r$.

Let $g_\Gamma : \Sigma_\Gamma \to \Sigma$ be the gluing map. We denote by $\widetilde{\Gamma}_i$ the boundary component of the cylinder $\Sigma_i$ that comes from the graph (that is $g_\Gamma(\widetilde{\Gamma}_i) \subset \Gamma$) and by $C_i$ the one coming from a boundary component of $\Sigma$ (that is $g_\Gamma(C_i) \subset \partial\Sigma$). From now on, we take the convention that $C_i$ is identified with $C_i \times \{1\}$ and that $\tilde{\Gamma}_i$ is identified with $C_i \times \{0\}$. We set $\Sigma_i := g_\Gamma(\widetilde{\Sigma}_i)$ and $\Gamma_i := g_\Gamma(\widetilde{\Gamma}_i)$. Finally we denote $g_\Gamma(C_i)$ also by $C_i$ since $g_\Gamma|_{C_i}$ is bijective.

A *retraction* or a *product structure* for a component $\Sigma_i$ is a parametrization

$$r_i : \mathbb{S}^1 \times I \to \Sigma_i.$$

For each $\theta \in \mathbb{S}^1$, we call $r_i(\{\theta\} \times I)$ a *retraction line.* We also say that $g_\Gamma(r_i(\{\theta\} \times I))$ is a retraction line.

**4.5.** A relative tête-à-tête graph $(\Gamma, A) \hookrightarrow (\Sigma, \partial A)$ induces a mapping class $[\phi_\Gamma]_{\partial\Sigma \setminus A, \mathrm{id}}$ on $\Sigma$, more specifically, an element of $\mathrm{MCG}(\Sigma, \partial\Sigma \setminus A)$. If a product structure is specified, then an explicit representative $\phi_\Gamma$ is induced. For any product structure $\phi_\Gamma$ satisfies:

1) $\phi_\Gamma|_\Gamma(p) = \gamma_p(\pi)$, that is, it induces on the graph the same action as the tête-à-tête property.
2) the mapping class $[\phi_\Gamma] \in \mathrm{MCG}(\Sigma)$ is of finite order, we also say it is periodic.
3) the fractional Dehn twist coefficients $t_i/n$ (recall 2.2) along all boundary components in $\Sigma \setminus A$ are strictly positive.

Actually, in [FPP] it is proven that 2) and 3) above characterize tête-à-tête automorphisms, more concretely the following is proven:

**Theorem 4.6.** *Let $\phi : \Sigma \to \Sigma$ be an automorphism of a surface with $\phi|_{\partial^1\Sigma} = \mathrm{id}$ for some non-empty subset $\partial^1\Sigma \subset \partial\Sigma$. Then there exists a relative tête-à-tête graph $(\Gamma, \partial\Sigma \setminus \partial^1\Sigma)$ with $[\phi_\Gamma]_{\partial^1\Sigma} = [\phi]_{\partial^1\Sigma}$ if and only if $[\phi] \in \mathrm{MCG}(\Sigma)$ is of finite order and all the fractional Dehn twists at boundary components in $\partial^1\Sigma$ are strictly positive.*

It corresponds to Theorem 9.12 in [FPP]. Actually it is a consequence of that Theorem since there, the authors consider also negative fractional Dehn twists and also *negative* safe walks (which turn left instead of turning right). This is done via a *sign* map $\iota : \partial^1\Sigma \to \{+, -, 0\}$.

In this work we only use the original notion of A'Campo, so this map is constant $+$.

## 5. Mixed tête-à-tête graphs

With pure, relative and general tête-à-tête graphs we model periodic automorphisms. Now we extend the notion of tête-à-tête graph to be able to model some pseudo-periodic automorphisms.

Let $(\Gamma^\bullet, A^\bullet)$ be a decreasing filtration on a connected relative metric ribbon graph $(\Gamma, A)$. That is

$$(\Gamma, A) = (\Gamma^0, A^0) \supset (\Gamma^1, A^1) \supset \cdots \supset (\Gamma^d, A^d)$$

where $\supset$ between pairs means $\Gamma^i \supset \Gamma^{i+1}$ and $A^i \supset A^{i+1}$, and where $(\Gamma^i, A^i)$ is a (possibly disconnected) relative metric ribbon graph for each $i = 0, \ldots, d$. We say that $d$ is the depth of the filtration $\Gamma^\bullet$. We assume each $\Gamma^i$ does not have univalent vertices and is a subgraph of $\Gamma$ in the usual terminology in Graph Theory. We observe that since each $(\Gamma^i, A^i)$ is a relative metric ribbon graph, we have that $A^i \setminus A^{i+1}$ is a disjoint union of connected components homeomorphic to $\mathbb{S}^1$.

For each $i = 0, \ldots, d$, let

$$\delta_i : \Gamma^i \to \mathbb{R}_{\geq 0}$$

be a locally constant map (so it is a map constant on each connected component). We put the restriction that $\delta_0(\Gamma^0) > 0$. We denote the collection of all these maps by $\delta_\bullet$.

Let $p \in \Gamma$, we define $c_p$ as the largest natural number such that $p \in \Gamma^{c_p}$.

**Definition 5.1** (Mixed safe walk)**.** Let $(\Gamma^\bullet, A^\bullet)$ be a filtered relative metric ribbon graph. Let $p \in \Gamma \setminus A \setminus v(\Gamma)$. We define a mixed safe walk $\gamma_p$ starting at $p$ as a concatenation of paths defined iteratively by the following properties

i) $\gamma_p^0$ is a safe walk of length $\delta_0(p)$ starting at $p_0^\gamma := p$. Let $p_1^\gamma := \gamma^0(\delta_0)$ be its endpoint.

ii) Suppose that $\gamma_p^{i-1}$ is defined and let $p_i^\gamma$ be its endpoint.
   - If $i > c_p$ or $p_i^\gamma \notin \Gamma^i$ we stop the algorithm.
   - If $i \leq c_p$ and $p_i^\gamma \in \Gamma^i$ then define $\gamma_p^i : [0, \delta_i(p_i)] \to \Gamma^i$ to be a safe walk of length $\delta_i(p_i^\gamma)$ starting at $p_i^\gamma$ and going in the same direction as $\gamma_p^{i-1}$.

iii) Repeat step *ii*) until algorithm stops.

Finally, define $\gamma_p := \gamma_p^k \star \cdots \star \gamma_p^0$, that is, the mixed safe walk starting at $p$ is the concatenation of all the safe walks defined in the inductive process above.

As in the pure case, there are two safe walks starting at each point on $\Gamma \setminus (A \cup v(\Gamma))$. We denote them by $\gamma_p$ and $\omega_p$.

**Definition 5.2** (Boundary mixed safe walk)**.** Let $(\Gamma^\bullet, A^\bullet)$ be a filtered relative metric ribbon graph and let $p \in A$. We define a boundary mixed safe walk $b_p$ starting at $p$ as a concatenation of a collection of paths defined iteratively by the following properties

i) $b_{p_0}^0$ is a boundary safe walk of length $\delta_0(p)$ starting at $p_0 := p$ and going in the direction indicated by $A$ (as in the relative tête-à-tête case). Let $p_1 := b_p^0(\delta_0)$ be its endpoint.

ii) Suppose that $b_{p_{i-1}}^{i-1}$ is defined and let $p_i$ be its endpoint.
   - If $i > c_p$ or $p_i \notin \Gamma^i$ we stop the algorithm.
   - If $i \leq c(p)$ and $p_i \in \Gamma^i$ then define $b_{p_i}^i : [0, \delta_i(p_i)] \to \Gamma^i$ to be a safe walk of length $\delta_i(p_i)$ starting at $p_i$ and going in the same direction as $b_{p_{i-1}}^{i-1}$.

iii) Repeat step *ii*) until algorithm stops.

Finally, define $b_p := b_{p_k}^k \star \cdots \star b_{p_0}^0$, that is, the boundary mixed safe walk starting at $p$ is the concatenation of all the safe walks defined in the inductive process.

**Notation 5.3.** We call the number $k$ in Definition 5.1 (resp. Definition 5.2), the *order* of the mixed safe walk (resp.boundary mixed safe walk) and denote it by $o(\gamma_p)$ (resp. $o(b_p)$).

We denote by $l(\gamma_p)$ the *length* of the mixed safe walk $\gamma_p$ which is the sum $\sum_{j=0}^{o(\gamma_p)} \delta_j(p_j^\gamma)$ of the lengths of all the walks involved. We consider the analogous definition $l(b_p)$.

As in the pure case, two mixed safe walks starting at $p \in \Gamma \setminus v(\Gamma)$ exist. We denote by $\omega_p$ the mixed safe walk that starts at $p$ but in the opposite direction to the starting direction of $\gamma_p$.

Observe that since the safe walk $b_{p_0}^0$ is completely determined by $p$, for a point in $A$ there exists only one boundary safe walk.

Now we define the relative mixed tête-à-tête property.

**Definition 5.4** (Relative mixed tête-à-tête property)**.** Let $(\Gamma^\bullet, A^\bullet)$ be a filtered relative metric ribbon graph and let $\delta_\bullet$ be a set of locally constant mappings $\delta_k : \Gamma^k \to \mathbb{R}_{\geq 0}$. We say that $(\Gamma^\bullet, A^\bullet, \delta_\bullet)$ satisfies the relative mixed tête-à-tête property or that it is a relative mixed tête-à-tête graph if for every $p \in \Gamma - (v(\Gamma) \cup A)$

I) The endpoints of $\gamma_p$ and $\omega_p$ coincide.
II) $c_{\gamma_p(l(\gamma_p))} = c_p$

and for every $p \in A$, we have that

III) $b_p(l(b_p)) \in A^{c_p}$

As a consequence of the two previous definitions we have:

**Lemma 5.5.** *Let $(\Gamma^\bullet, A^\bullet, \delta_\bullet)$ be a mixed relative tête-à-tête graph, then*

a) $o(\omega_p) = o(\gamma_p) = c_p$
b) $l(\gamma_p) = l(\omega_p)$ *for every* $p \in \Gamma \backslash v(\Gamma)$.

*Proof.* See [FPP]. □

**Remark 5.6.** Note that for mixed tête-à-tête graphs it is not true that $p \mapsto \gamma_p(\delta(p))$ gives a continuous mapping from $\Gamma$ to $\Gamma$.

**Remark 5.7.** Satisfying $I$) and $II$) of the mixed tête-à-tête property in Definition 5.4 is equivalent to satisfying:

I') For all $i = 0, \ldots, d-1$, the automorphism $\widetilde{\phi}_{\Gamma,i} = \mathcal{D}_{\delta_i} \circ \phi_{\Gamma,i-1}$ is compatible with the gluing $g_i$, that is,

$$g_i(x) = g_i(y) \Rightarrow g_i(\widetilde{\phi}_{\Gamma,i}(x)) = g_i(\widetilde{\phi}_{\Gamma,i}(y)).$$

Below we see the diagram which shows the construction of $\phi_\Gamma$.

(5.8)

$$\begin{tikzcd}
\Sigma_{\Gamma^0} \arrow[r, "\phi_{\Gamma,-1}"] & \Sigma_{\Gamma^0} \arrow[r, "\mathcal{D}_{\delta_0}"] & \Sigma_{\Gamma^0} \arrow[d, "g_0"] & & & & & & \\
\Sigma_{\Gamma^1} \arrow[u, leftarrow, "g_0"'] \arrow[rr, "\phi_{\Gamma,0}"] & & \Sigma_{\Gamma^1} \arrow[r, "\mathcal{D}_{\delta_1}"] & \Sigma_{\Gamma^1} \arrow[d, "g_1"] & & & & & \\
\Sigma_{\Gamma^2} \arrow[u, leftarrow, "g_1"'] \arrow[rrr, "\phi_{\Gamma,1}"] \arrow[d, "g_2"] & & & \Sigma_{\Gamma^2} \arrow[r, "\mathcal{D}_{\delta_2}"] & \Sigma_{\Gamma^2} \arrow[d, "g_2"] & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & & \\
\Sigma_{\Gamma^d} \arrow[rrrrrrr, "\phi_{\Gamma,d-1}"] \arrow[d, "g_d"] & & & & & & & \Sigma_{\Gamma^d} \arrow[r, "\mathcal{D}_{\delta_d}"] & \Sigma_{\Gamma^d} \arrow[d, "g_d"] \\
\Sigma \arrow[rrrrrrrr, "\phi_{\Gamma}=\phi_{\Gamma,d}"] & & & & & & & & \Sigma
\end{tikzcd}$$

**Remark 5.9.** The pseudo-periodic automorphism $\phi_\Gamma$ induced by a mixed tête-à-tête graph has negative screw numbers and positive fractional Dehn twist coefficients as noted in [FPP]. Actually, it was also proved in [FPP] that the screw number associated to an orbit of annuli $\mathcal{A}^i_{j,1}, \dots, \mathcal{A}^i_{j,k}$ between levels $i-1$ and $i$ of the filtration is

$$-\sum_k \delta_i(\Gamma^i_{j,k})/l(\tilde{\Gamma}^i_{j,1}). \tag{5.10}$$

## 6. Realization theorem

In this section we prove Theorem 6.7 which is the main results of this paper. It characterizes the pseudo-periodic automorphisms that can be realized by mixed tête-à-tête graphs. First we introduce some notation and conventions.

Let $\phi : \Sigma \to \Sigma$ be a pseudo-periodic automorphism. For the remaining of this work we impose the following restrictions on $\phi$:

(i) The screw numbers are all negative.
(ii) It leaves at least one boundary component pointwise fixed and the fractional Dehn twist coefficients at these boundary components are positive.

Denote by $\partial^1\Sigma \subset \partial\Sigma$ the union of the boundary components pointwise fixed by $\phi$. We assume that $\phi$ is given in some almost-canonical form as in Remark 3.3.

**Notation 6.1.** We define a graph $G(\phi)$ associated to a given almost-canonical form:

(i) It has a vertex $v$ for each subsurface of $\Sigma \backslash \mathcal{A}$ whose connected components are cyclically permuted by $\phi$.
(ii) For each set of annuli in $\mathcal{A}$ permuted cyclically it has an edge connecting the vertices corresponding to the surfaces on each side of the collection of annuli.

Let $\mathcal{N}$ denote the set of vertices of $G(\phi)$.

**Definition 6.2.** We say that a function $L : \mathcal{N} \to \mathbb{Z}_{\geq 0}$ is a *filtering function* for $G(\phi)$ if it satisfies:

(i) If $v, v' \in \mathcal{N}$ are connected by an edge, then $L(v) \neq L(v')$.
(ii) If $v \in \mathcal{N}$, then either $v$ has a neighbor $v' \in \mathcal{N}$ with $L(v) > L(v')$, or $L(v) = 0$ and $\Sigma_v$ contains a component of $\partial^1\Sigma$.

Condition $ii$) above implies that for $L$ to be a filtering function, $L^{-1}(0)$ must only contain vertices corresponding to subsurfaces of $\Sigma \backslash \mathcal{A}$ that contain some component of $\partial^1\Sigma$. That same condition assures us that $L^{-1}(0)$ is non-empty.

**Definition 6.3.** Define the distance function $D : \mathcal{N} \to \mathbb{Z}_{\geq 0}$ as follows:

(i) $D(v) = 0$ for all $v$ with $\Sigma_v \cap \partial^1\Sigma \neq \emptyset$.
(ii) $D(v)$ is the distance to the set $D^{-1}(0)$, that is the number of edges of the smallest bamboo in $G(\phi)$ connecting $v$ with some vertex in $D^{-1}(0)$.

**Remark 6.4.** Take some $\phi : \Sigma \to \Sigma$ in canonical form and observe that the function $D$ might not be a filtering function. It can happen that there are two adjacent vertices $v, v' \in \mathcal{N}$ with $D(v) = D(v')$ or even that there is a vertex with a loop based at it (see Example 6.13). However we modify the canonical form into an almost-canonical form for which the function $D$ *is* a filtering function:

Let $\phi : \Sigma \to \Sigma$ be an automorphism in canonical form such that $D(v) = D(v')$ for some adjacent $v, v' \in \mathcal{N}$. Take one edge joining $v$ and $v'$, this edges corresponds to a set of annuli $\mathcal{A}_1, \ldots, \mathcal{A}_k$ being permuted cyclically by $\phi$. For each $i = 1, \ldots, k$, let $\eta_i : \mathbb{S}^1 \to \mathcal{A}_i$ be parametrizations as in Lemma 3.16. Let $\mathcal{C}_i \subset \mathcal{A}_i$ be the core curves of the annuli. We distinguish two cases:

1) The core curves are not amphidrome. By Remark 3.6 we can isotope $\phi$ on the annuli $\mathcal{A}_i$ to a automorphism $\tilde{\phi}$ without changing the action of $\phi$ on $\partial\mathcal{A}_i$ so that in the annuli $\eta_i(\mathbb{S}^1 \times [\frac{1}{3}, \frac{2}{3}])$ it is periodic. In doing so, we can redefine the canonical form to an almost-canonical form as follows.
   (a) for each $i = 1, \ldots, k$ take $\mathcal{C}_i$ out from the set $\mathcal{C}$ and include $\eta_i(\mathbb{S}^1 \times \{\frac{1}{6}\})$ and $\eta_i(\mathbb{S}^1 \times \{\frac{5}{6}\})$.
   (b) for each $i = 1, \ldots, k$ take $\mathcal{A}_i$ out of $\mathcal{A}$ and include $\eta_i(\mathbb{S}^1 \times [0, 1/3])$ and $\eta_i(\mathbb{S}^1 \times [\frac{2}{3}, 1])$.

   It is clear that this new set of data defines an almost canonical form for $\tilde{\phi}$ and that on the corresponding $G(\tilde{\phi})$ the vertices $v$ and $v'$ are no longer adjacent since a new vertex corresponding to the surface $\bigcup_i \eta_i(\mathbb{S}^1 \times [\frac{1}{3}, \frac{2}{3}])$ appears between them.
2) The core curves are amphidrome. This case is completely analogous to case 1) with the advantage that by definition of $\tilde{\mathcal{D}}_s$ inNotation 3.8, it is already periodic in the central annuli.

It is clear that after performing 1) or 2) (accordingly) for all pairs of adjacent vertices $v, v'$ with $D(v) = D(v')$ we provide $\phi$ with an almost-canonical whose distance function $D$ is a filtering function.

**Remark 6.5.** We observe that orbits of amphidrome annuli $\mathcal{A}_1, \ldots, \mathcal{A}_i$ correspond to loops in $G(\phi)$. So we have that after performing the modification of Remark 6.4, the almost-canonical form of $\phi$ does not have any amphidrome annuli in $\mathcal{A}$. However, some of the surfaces of $\Sigma \setminus \mathcal{A}$ are now amphidrome annuli.

**Notation 6.6.** We assume $\phi$ is in the almost-canonical form induced from the canonical form after performing the modification described in Remark 6.4. We denote by $\hat{\Sigma}$ the closure of $\Sigma \setminus \mathcal{A}$ in $\Sigma$. Let $\hat{G}(\phi)$ be a graph constructed as follows:

(i) It has a vertex for each connected component of $\Sigma \setminus \mathcal{C}$.
(ii) There are as many edges joining two vertices as curves in $\mathcal{C}$ intersect the two surfaces corresponding to those vertices.

We observe that the previously defined $G(\phi)$ is nothing but the quotient of $\hat{G}(\phi)$ by the action induced by $\phi$ on the connected components of $\Sigma \setminus \mathcal{A}$.

Let $\hat{\mathcal{N}}$ be the set of vertices of $\hat{G}(\phi)$. Since $\phi$ permutes the surfaces in $\hat{\Sigma}$, it induces a permutation of the set $\hat{\mathcal{N}}$ which we denote by $\sigma_\phi$. We label the set $\hat{\mathcal{N}}$, as well as the connected components of $\hat{\Sigma}$ and the connected components of $\mathcal{A}$ in the following way:

(i) Label the vertices that correspond to surfaces containing components of $\partial^1\Sigma$ by $v^0_{1,1}, v^0_{2,1}, \ldots, v^0_{\beta_0,1}$. Let $V^0$ be the union of these vertices. Note that $\sigma_\phi(v^0_{j,1}) = v^0_{j,1}$ for all $j = 1, \ldots, \beta_0$.
(ii) Let $\hat{D} : \hat{\mathcal{N}} \to \mathbb{Z}_{\geq 0}$ be the distance function to $V^0$, that is, $\hat{D}(v)$ is the number of edges of the smallest path in $\hat{G}(\phi)$ that joins $v$ with $V^0$. Let $V^i := \hat{D}^{-1}(i)$. Observe that the permutation $\sigma_\phi$ leaves the set $V^i$ invariant. There is a labeling of $V^i$ induced by the orbits of $\sigma_\phi$: suppose it has $\beta_i$ different orbits. For each $j = 1, \ldots, \beta_i$, we label the vertices in that orbit by $v^i_{j,k}$ with $k = 1, \ldots, \alpha_j$ so that $\sigma_\phi(v^i_{j,k}) = v^i_{j,k+1}$ and $\sigma_\phi(v^i_{j,\alpha_j}) = v^i_{j,1}$.

Denote by $\Sigma^i_{j,k}$ the surface in $\hat{\Sigma}$ corresponding to the vertex $v^i_{j,k}$. Denote by $\Sigma^i$ the union of the surfaces corresponding to the vertices in $V^i$. We denote by $\Sigma^{\leq i}$ the union of $\Sigma^0, \ldots, \Sigma^i$ and the annuli in between them.

We recall that $\alpha_j$ is the smallest positive number such that $\phi^{\alpha_j}(\Sigma^i_{j,k}) = \Sigma^i_{j,k}$.

**Theorem 6.7.** *Let $\phi : \Sigma \to \Sigma$ be a pseudo-periodic automorphism satisfying assumptions (i) and (ii). Then there exists a relative mixed tête-à-tête graph $(\Gamma^\bullet, A^\bullet, \delta_\bullet)$ with $\Gamma$ embedded in $\Sigma$ such that:*

*(i)* $\delta_i$ *is a constant function for each* $i = 1, \ldots, d$.
*(ii)* $[\phi]_{\partial^1\Sigma} = [\phi_\Gamma]_{\partial^1\Sigma}$.
*(iii)* $\phi|_{\partial\Sigma\setminus\partial^1\Sigma} = \phi_\Gamma|_{\partial\Sigma\setminus\partial^1\Sigma}$.
*(iv) Filtration indexes are induced by the distance function $D$ for the almost-canonical form induced from the canonical form by Remark 6.4.*

Now we state and prove Lemma 6.8 and Lemma 6.9 which are used in the proof of Theorem 6.7.

**Lemma 6.8.** *Let $\phi : \Sigma \to \Sigma$ be a periodic automorphism of order $n$. Let $C = C_1 \sqcup \cdots \sqcup C_k$ be a non-empty collection of boundary components of $\Sigma$ such that $\phi(C_i) = C_i$, that is, each one is invariant by $\phi$. For each $i$ let $m_i$ be a metric on $C_i$ invariant by $\phi$. Then there exists a relative metric ribbon graph $(\Gamma, A) \hookrightarrow (\Sigma, \partial\Sigma\backslash C)$ and parametrizations of the cylinders (see Notation 4.4) $r_i : \mathbb{S}^1 \times I \to \tilde{\Sigma}_i$ such that:*

*(i)* $\phi(\Gamma) = \Gamma$ *and the metric of* $\Gamma$ *is also invariant by* $\phi$.
*(ii)* $l(\tilde{\Gamma}_i) = l(C_i)$.
*(iii) The projection from $C_i$ to $\tilde{\Gamma}_i$ induced by $r_i$ is an isometry, that is, the map*

$$r(\theta, 0) \mapsto r(\theta, 1)$$

*is an isometry.*

*(iv)* $\phi$ *sends retractions lines (i.e.* $\{\theta\} \times I$*) to retractions lines.*

*Proof.* The proof uses essentially the same technique used in the proof of Theorem 5.4 and 6.2 in [FPP]. For completeness we outline it here.

Let $\Sigma^\phi$ be the orbit surface and suppose it has genus $g$ and $r \geq k$ boundary components. Let $p : \Sigma \to \Sigma^\phi$ be the induced branch cover.

Take any relative spine $\Gamma^\phi$ of $\Sigma^\phi$ that:

1) Contains all branch points of the map $p$.
2) Contains the boundary components $p(\partial\Sigma \setminus C)$.
3) Admits a metric such that $p(C_i)$ retracts to a part of the graph of length $l(C_i)/n$.

We observe that conditions 1) are 2) are trivial to get. Condition 3) follows because of the proof of Theorem 5.4 and 6.2 in [FPP]. There, the conditions on the metric of the graph $\Gamma^\phi$ come from the rotation numbers of $\phi$, however, we do not use that these numbers come from $\phi$ in finding the appropriate graph so exactly the same argument applies.

Observe that since the metric on $C_i$ is invariant by $\phi$, there is a metric induced on $p(C_i)$ for $i = 1, \dots k$. Now choose any parametrizations (or product structures) of the cylinders in $\Sigma^\phi_{\Gamma^\phi}$ such that their retractions lines induce an isometry from $p(C_i)$ to $\tilde{\Gamma}^\phi_i$.

Define $\Gamma := p^{-1}(\Gamma^\phi)$. By construction, this graph satisfies $(i)$ and $(ii)$. The preimage by $p$ of retraction lines on $\Sigma^\phi$ gives rise to parametrizations of the cylinders in $\Sigma_\Gamma$ satisfying $iii)$ and $iv)$. □

**Lemma 6.9.** *Let* $(\Gamma^\bullet, B^\bullet, \delta_\bullet)$ *be a relative mixed tête-à-tête graph embedded in a surface* $\Sigma$ *and let* $C_1, \dots, C_k \subset B$ *a set of relative boundary components cyclically permuted by* $\phi$*. Suppose that all the vertices in these boundary components are of valency* 3*. Then we can modify the metric structure of the graph to produce a mixed tête-à-tête graph* $(\hat{\Gamma}^\bullet, \hat{B}^\bullet, \delta_\bullet)$ *with* $l(C_i)$ *as small as we want and with* $[\phi_\Gamma]_{\partial^1\Sigma} = [\phi_{\hat{\Gamma}}]_{\partial^1\Sigma}$.

*Proof.* Let $e_1, \dots, e_m$ be the edges comprising $C_1$, where $e_j$ has length $l_j$. Let $v_1, \dots, v_m$ be the vertices of these edges, so that $e_i$ connects $v_i$ and $v_{i+1}$ (here, indices are taken modulo $m$). Let $f_{i,j}$, for $j = 1, \dots, n_i$ be the edges adjacent to $v_i$, other than $e_i, e_{i+1}$, in such a way that the edges have the cyclic order $e_{i+1}, e_i, f_{i,1}, \dots, f_{i,n_i}$. Let $\varepsilon < l(C_1)$. We would like to replace $C_1$ with a circle of length $l(C_1) - \varepsilon$. We assume that $l_1 = \min_i l_i$.

If $\varepsilon/m \leq l_1$, then we do the following:

- ❀ Each edge $e_i$ is modified to have length $l_i - \varepsilon/m$.
- ❀ For any $i$ with $n_i = 1$, the length of $f_{i,1}$ is increased by $\varepsilon/2m$.
- ❀ For any $i$ with $n_i > 1$, extrude an edge $g_i$ from the vertex $v_i$ of length $\varepsilon/2m$ so that one end of $g_i$ is adjacent to $e_i, e_{i+1}$ and $g_i$, and the other is adjacent to $f_{i,1}, \dots, f_{i,n_i}$ and $g_i$, with these cyclic orders.

In the case when $\varepsilon/m > l_1$, we execute the above procedure with $\varepsilon$ replaced by $m \cdot l_1$, which results in a circle made up of fewer edges. After finitely many steps, we obtain the desired length for $C_1$.

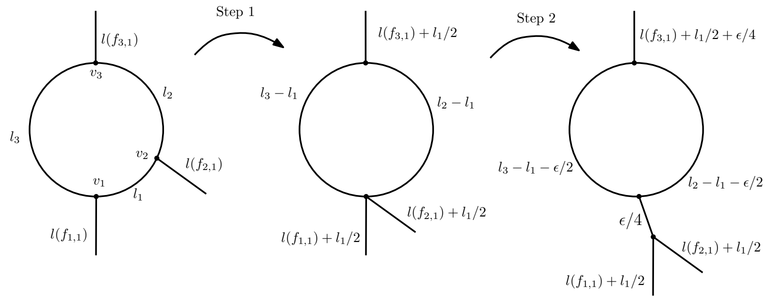


**Figure 6.1.** Example of modification at a boundary component. Suppose that $l_1 < l_2 < l_3$. In step 1 we reduce the length of the circle by $l_1$. In step 2 we reduce it by $\varepsilon$.

□

*Proof of Theorem 6.7.* By definition, all surfaces corresponding to vertices in $D^{-1}(0)$ are connected because they are invariant by $\phi$. We have that $\phi|_{\Sigma^0_{j,1}} : \Sigma^0_{j,1} \to \Sigma^0_{j,1}$ is periodic outside a neighborhood of $\partial^1\Sigma \cap \Sigma^0_{j,1}$ and that the fractional Dehn twist coefficients with respect to all the components in $\partial^1\Sigma \cap \Sigma^0_{j,1}$ are positive. Denote $B^0_{j,1} := \partial\Sigma^0_{j,1} \setminus \partial^1\Sigma$. By Theorem 4.6, for each $j = 1, \dots, \beta_0$, there is a relative tête-à-tête graph $(\Gamma^0_{j,1}, B^0_{j,1})$ embedded in $\Sigma^0_{j,1}$ modeling $\phi|_{\Gamma^0_{j,1}}$. Denote $\Gamma[0] := \bigsqcup_j \Gamma^0_{j,1}$ and $B[0] := \bigsqcup_j B^0_{j,1}$. Then $(\Gamma[0], B[0], \delta_0)$ is a relative mixed tête-à-tête graph of depth 0 for $\Sigma^{\leq 0}$ (it is just a relative tête-à-tête graph) such that:

(i) $\delta_0|_{\Sigma^0_{j,1}} = \pi$ for all $j = 1, \dots, \beta_0$.
(ii) $[\phi|_{\Sigma^0}]_{\partial^1\Sigma^0} = [\phi_{\Gamma[0]}]_{\partial^1\Sigma^0}$.
(iii) $\phi|_{B[0]} = \phi_{\Gamma[0]}|_{B[0]}$.
(iv) All the vertices on $B[0]$ have valency 3.

Suppose that we have a relative mixed tête-à-tête graph $(\Gamma[a-1]^\bullet, B[a-1]^\bullet, \delta[a-1]_\bullet)$ of depth $a$ embedded as a spine in $\Sigma^{\leq a}$ and with $B[a-1] = \partial\Sigma^{\leq a-1} \setminus \partial^1\Sigma$ such that:

(i) $\delta[a-1]_i$ is a constant function for each $i = 0, \dots, a-1$
(ii) $[\phi|_{\Sigma^{\leq a-1}}]_{\partial^1\Sigma^{\leq a-1}} = [\phi_{\Gamma[a-1]}]_{\partial^1\Sigma^{\leq a-1}}$
(iii) $\phi|_{B[a-1]} = \phi_{\Gamma[a-1]}|_{B[a-1]}$.
(iv) All the vertices on $B[a-1]$ have valency 3.

We recall that $\phi_{\Gamma[a-1]}$ denotes the mixed tête-à-tête automorphism induced by $(\Gamma[a-1]^\bullet, B[a-1]^\bullet, \delta[a-1]_\bullet)$. We extend $\Gamma[a-1]$ to a mixed tête-à-tête graph $\Gamma[a]$ satisfying (i) - (iv). This proves the theorem by induction. We focus on a particular orbit of surfaces. Fix $j \in \{1, \dots, \beta_a\}$ and consider the surfaces $\Sigma^a_{j,1}, \dots, \Sigma^a_{j,\alpha_j} \subset \Sigma^a$ with $\phi(\Sigma^a_{j,k}) = \Sigma^a_{j,k+1}$ and $\phi(\Sigma^a_{j,\alpha_j}) = \Sigma^a_{j,1}$.

For each $j$, we distinguish two types of boundary components in the orbit $\bigsqcup_k \Sigma^a_{j,k}$:

Type I) Boundary components that are connected to an annulus whose other end is in $\Sigma^{a-1}$, we denote these by $\partial^I$.
Type II) The rest: boundary components that are in $\partial\Sigma$ and boundary components that are connected to an annulus whose other end is in $\Sigma^{a+1}$, we denote these by $\partial^{II}$.

Since we are doing the construction for an orbit, we use *local notation* in which not all the indices are specified so that the formulae is easier to read.

Let $\mathcal{A}^I$ denote the union of annuli connected to boundary components in $\partial^I$. These annuli are permuted by $\phi$. Suppose that there are $r'$ different orbits of annuli $\mathcal{A}_1, \ldots \mathcal{A}_{r'}$, and let $\ell_i \in \mathbb{N}$ be the length of the orbit $\mathcal{A}_i$. Let $s_i$ be the screw number of the orbit $\mathcal{A}_i$ (recall Definition 3.14 and Remark 3.15). Let $\mathcal{B}_{i,1}, \ldots, \mathcal{B}_{i,\ell_i}$ be the orbit of boundary components of $\Sigma^{a-1}$ that are contained in the orbit $\mathcal{A}_i$. The metric of $\Gamma[a-1]$ gives lengths to these boundary components and all the boundary components in the same orbit have the same length $l(\mathcal{B}_{i,1}) \in \mathbb{R}_+$. Consider the positive real numbers

$$\frac{s_1}{\ell_1} l(\mathcal{B}_{1,1}), \ldots, \frac{s_{r'}}{\ell_{r'}} l(\mathcal{B}_{r',1}) \tag{6.10}$$

Using Lemma 6.9, we modify the metric structure of $\Gamma[a-1]$ near each orbit $\mathcal{B}_i$ so that

$$\frac{s_1}{\ell_1} l(\mathcal{B}_{1,1}) = \cdots = \frac{s_{r'}}{\ell_{r'}} l(\mathcal{B}_{r',1}).$$

This is possible since we can make $l(\mathcal{B}_{i,1})$ as small as needed.

For each $i = 1, \ldots, r'$, let $\mathcal{A}_{i,1}, \ldots \mathcal{A}_{i,\ell_i}$ be the annuli in the orbit $\mathcal{A}_i$ and let $\mathcal{B}'_{i,1}, \ldots \mathcal{B}'_{i,\ell_i}$ be the boundary components that they share with $\Sigma^a$. Consider parametrizations $\eta_{i,1}, \ldots, \eta_{\ell_i,1}$ given by Lemma 3.16. The metric on the boundary components of $B[a-1]$ and the parametrizations induce a metric on all the boundary components in $\partial^I$ that is invariant by $\phi$.

We observe that $\phi^{\alpha_j}|_{\Sigma^a_{j,1}} : \Sigma^a_{j,1} \to \Sigma^a_{j,1}$ is periodic and $\partial^I \cap \Sigma^a_{j,1}$ is a subset of boundary components that have a metric. So we can apply Lemma 6.8 and we get a relative metric ribbon graph $(\Gamma^a_{j,1}, \partial^{II} \cap \Gamma^a_{j,1})$ and parametrizations of each cylinder in $(\Sigma^a_{j,1})_{\Gamma^a_{j,1}}$ with properties $i), \ldots, iv)$ in the the Lemma. We can translate this construction by $\phi$ to the rest of the surfaces $\Sigma^a_{j,2}, \ldots, \Sigma^a_{j,\alpha_j}$. So we get graphs $\Gamma^a_{j,k} \hookrightarrow \Sigma^a_{j,k}$ and parametrizations for the cylinders in $(\Sigma^a_{j,k})_{\Gamma^a_{j,k}}$ for all $k = 1, \ldots, \alpha_j$. The construction assures us that $\phi|_{\Sigma^a_{j,\alpha_j}} : \Sigma^a_{j,\alpha_j} \to \Sigma^a_{j,1}$ sends $\Gamma^a_{j,\alpha_j}$ to $\Gamma^a_{j,1}$ isometrically and that it takes retractions lines of the parametrizations in $\Sigma^a_{j,\alpha_j}$ to retraction lines in $\Sigma^a_{j,1}$.

We proceed to extend $\Gamma[a-1]$ to the orbit of $\Sigma^a_{j,1}$. For each $i = 1, \ldots, r'$ do the following:

Step 1. Remove $\mathcal{B}_{i,1}, \ldots, \mathcal{B}_{i,\ell_i}$ from $\Gamma[a-1]$.

Step 2. Take $\varepsilon > 0$ small enough. Decrease by $\varepsilon$ the metric on all the edges of $\Gamma[a-1]$ adjacent to vertices in $\mathcal{B}_{i,1}, \ldots, \mathcal{B}_{i,\ell_i}$.

Step 3. Add to the graph the retraction lines of the parametrizations $\eta_{i,1}, \ldots, \eta_{i,\ell_i}$ that were adjacent to vertices in $\mathcal{B}_{i,1}, \ldots, \mathcal{B}_{i,\ell_i}$. That is, if $v \in \mathcal{B}_{i,1} \subset \mathcal{A}_{i,1}$ include $\eta_{i,1}(\{v\} \times I)$. Define the length of these segments as $\varepsilon/2$.

Step 4. Add to the graph the retraction lines of the parametrizations of the cylinders $(\Sigma^a_{j,k})_{\Gamma^a_{j,k}}$ that start at the ends of the lines added in the previous step. Define the length of these segments as $\varepsilon/2$.

Step 5. Add to the graph the graphs $\Gamma^a_{j,1}, \ldots, \Gamma^a_{j,\alpha_j}$.

We repeat this process for all orbits of surfaces in $\Sigma^a$ and so we extend the graph $\Gamma[a-1]$ to all $\Sigma^a$. Denote

$$\Gamma[a]^a := \bigsqcup_{j,k} \Gamma^a_{j,k}.$$

Denote the resulting graph by $\Gamma[a]$.

We make the following observation: $(\Gamma[a]_{\Gamma[a]^a}, \tilde{\Gamma}[a]^a)$ is by construction isometric to $(\Gamma[a-1], \mathcal{B}[a-1])$. We denote the induced relative mixed tête-à-tête automorphism by $\phi_{\Gamma[a]_{\Gamma[a]^a}}$ which acts on $\Sigma^{\leq a}_{\Gamma[a]^a}$. By the previous observation there is an induced filtration on $\Gamma[a]$:

$$\Gamma[a] = \Gamma[a]^0 \supset \Gamma[a]^1 \supset \cdots \supset \Gamma[a]^{a-1} \supset \Gamma[a]^a$$

and similarly for the relative parts. We define $\delta_a : \Gamma[a]^a \to \mathbb{R}_{\geq}0$ to be the constant function equal to the numbers eq. (6.10). (which are by construction the same number).

By the choice of $\delta_a$ and the parametrizations on the annuli that join $\Sigma^{a-1}$ with $\Sigma^a$ we have that

$$\mathcal{D}_{\delta_a} \circ \phi_{\Gamma_{\Gamma[a]}} : \Sigma_{\Gamma[a]^a} \to \Sigma_{\Gamma[a]^a}$$

is compatible with the gluing $g_{a+1}$. So that $(\Gamma[a], \mathcal{B}[a])$ is a relative mixed tête-à-tête graph follows from $I'$) in Remark 5.7.

We have already made sure in the construction that (i) and (iv) hold in $\Gamma[a]$.

Let's show that (ii) and (iii) also hold. Observe that by construction $\phi$ leaves $\Gamma[a]$ invariant so there is an automorphism $\tilde{\phi}_a : \Sigma_{\Gamma[a]^a} \to \Sigma_{\Gamma[a]^a}$ induced. This automorphism coincides with $\mathcal{D}_{\delta_a} \circ \phi_{\Gamma_{\Gamma[a]}}$ on $\tilde{\Gamma}[a]$ by the choice of the parametrizations of the annuli $\mathcal{A}$ and by the choice of the number $\delta_a$. Also, by the choice of $\delta_a$ and [FPP, Remark 8.12] we see that they have the same screw numbers on the annuli connecting the level $a-1$ and the level $a$. From this discussion we get (ii) and (iii) and finish the proof.

□

**Remark 6.11.** From the proof we get as an important consequence that a more restrictive definition of a mixed tête-à-tête graph is valid: it is enough to consider mixed tête-à-tête graphs where $\delta_i$ is a constant function (i.e. a number) for all $i = 0, \ldots, \ell$.

**Corollary 6.12.** *The monodromy associated with a reduced holomorphic function germ defined on an isolated surface singularity is a mixed tête-à-tête twist. Conversely, let $C(\Gamma)$ be the cone over the open book associated with a mixed tête-à-tête graph. Then there exists a complex structure on $C(\Gamma)$ and a reduced holomorphic function germ $f : C(\Gamma) \to \mathbb{C}$ inducing $\phi_\Gamma$ as the monodromy of its Milnor fibration.*

*Proof.* The statement follows from Theorem 6.7 and [NP07, Theorem 2.1]. □

**Example 6.13.** Let $\Sigma$ be the surface of Figure 6.2. Suppose it is embedded in $\mathbb{R}^3$ with its boundary component being the unit circle in the $xy$-plane. Consider the rotation of $\pi$ radians around the $z$-axis and denote it by $R_\pi$. By the symmetric embedding of the surface, it leaves the surface invariant. Isotope the rotation so that it is the identity on $z \leq 0$ and it has fractional Dehn twist coefficient equal to 1/2. We denote the isotoped automorphism by $T$.

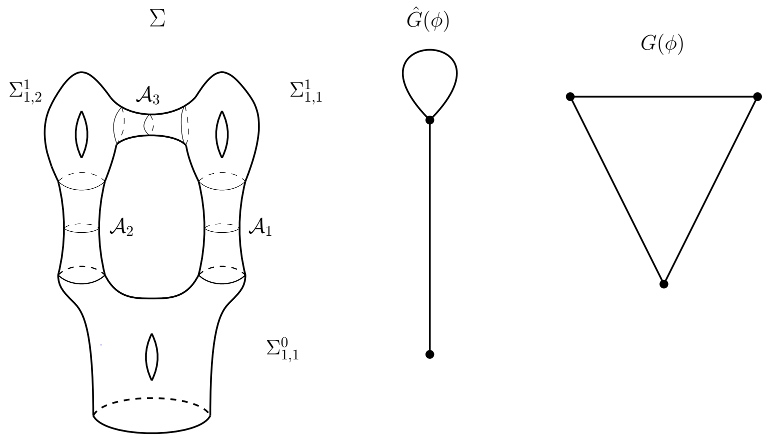


**Figure 6.2.** On the left we see the surface $\Sigma$. On the right we see the corresponding graphs $\hat{G}(\phi)$ and $G(\phi)$ for the depicted canonical form.

More concretely, let $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(r,\theta,z) := \begin{cases} (re^{i(\theta+\pi)}, z) & \text{if} \quad z \leq \varepsilon \\ (re^{\theta+\frac{z}{\varepsilon}\pi}, z) & \text{if} \quad 0 \leq z \leq \varepsilon \\ id & \text{if} \quad z \leq 0 \end{cases} \tag{6.14}$$

With $(r,\theta)$ polar coordinates on the $xy$-plane and $\varepsilon > 0$ small.

Let $D_i$ be a full positive Dehn twist on the annuli $\mathcal{A}_i$, $k = 1, 2, 3$ (See Figure 6.2). We define the automorphism

$$\phi := \mathcal{D}_3^{-1} \circ \mathcal{D}_2^{-1} \circ T|_\Sigma.$$

The automorphism comes already in canonical form. We construct the corresponding Nielsen graph $\hat{G}(\phi)$ and we observe that the corresponding distance function $D$ is not a filtering function since there is 1 loop on $\hat{G}(\phi)$. So we apply Remark 6.4 and we get the almost-canonical form and graphs of figure Figure 6.3.

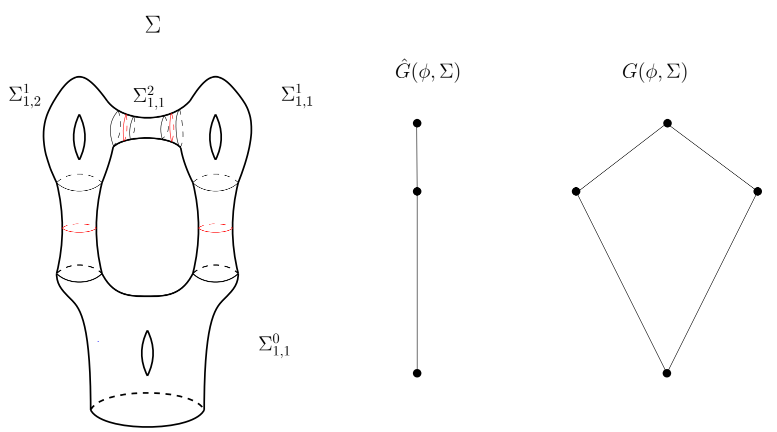


**Figure 6.3.** On the left we see the surface $\Sigma$. On the right we see the corresponding graphs $\hat{G}(\phi)$ and $G(\phi)$ for the depicted almost-canonical form. In red we see the core curves of the annuli in $\mathcal{A}$.

Now there are 4 annuli in $\mathcal{A}$ in this almost-canonical form. The annuli $\mathcal{A}_1$ and $\mathcal{A}_2$ are exchanged by the monodromy, the Dehn twists $\mathcal{D}_1^{-2}$ and $\mathcal{D}_2$ indicate that the screw number of this orbit is $-1$. And the annuli $\mathcal{A}_{3,1}$ and $\mathcal{A}_{3,2}$ that were originally contained in $\mathcal{A}_3$; these annuli are also exchanged by the monodromy. We get that the screw number on this orbit is $-1$.

We start the construction process following Theorem 6.7. We construct a relative tête-à-tête graph $(\Gamma[0], B[0])$ for $\phi|_{\Sigma^0} : \Sigma^0 \to \Sigma^0$. We use [FPP, Theorem 5.22] for this. In Figure 6.4 we can see this graph in blue.

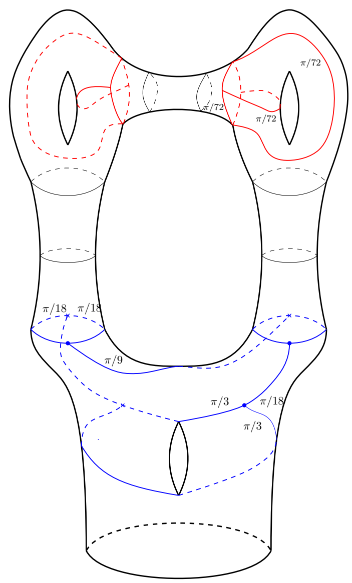


**Figure 6.4.** The relative tête-à-tête graph $(\Gamma[0], \mathcal{B}[0])$ embedded in $\Sigma^0 \subset \Sigma$. The lengths are indicated on a few edges and the rest is obtained by symmetry of the graph. We can also see in red an invariant relative spine for $\Sigma^1$.

In the next step we construct relative metric ribbon graphs for $\Sigma^1$. In this case $\Sigma^1$ consist of two connected surfaces that are exchanged. Each surface is a torus with two disks removed and one of the boundary components is glued to an annulus connecting it with $\Sigma^2$. This graph will correspond with $\Gamma^1_{\Gamma^2}$ in the final mixed tête-à-tête graph. In the notation of the Theorem we are using, it is $\Gamma[1]^1$. In Figure 6.4 we can see these relative metric ribbon graphs in red. In this step we also choose an invariant product structure on $\Sigma^1_{\Gamma^1}$

Now we proceed to find the parametrizations $\eta_1$ and $\eta_2$. We pick any parametrization $\eta_1$ for $\mathcal{A}_1$. On the right part of Figure 6.5 we can see the two retraction lines of the parametrization starting at the two vertices $p$ and $q$ of the corresponding boundary component in $B[0]$. On the left part of that figure we see two annuli, the upper one shows the image of the two retraction lines by $\phi$, on the lower annulus we see the retraction lines that we choose according to Lemma 3.16.

We concatenate the chosen retraction lines of the annuli (green in 6.6) with the corresponding retraction lines of the product structure in $\Sigma^1$ (orange in the the picture).

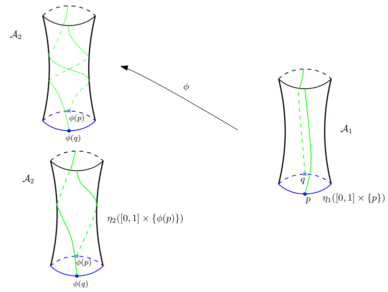


**Figure 6.5.** The orbit of annuli $\mathcal{A}_1$ and $\mathcal{A}_2$. On the right part we see $\mathcal{A}_1$ with a chosen product structure and on the left part we see two copies of $\mathcal{A}_2$, the lower one with the parametrization given by Lemma 3.16.

The metric on the red part is chosen so that each of the two components of $\tilde{\Gamma}^1$ has the same length as the two relative components in $\mathcal{B}[0]$, that is equal to $\pi/9$. Since the screw number is $-1$ and there are two annuli in the orbit we get that $\delta_1$ is the constant function $\pi/18$.

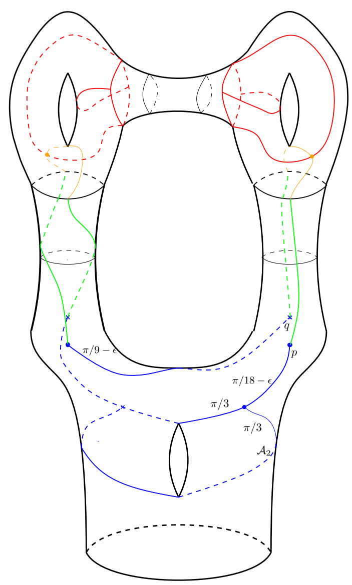


**Figure 6.6.** The relative tête-à-tête graph $(\Gamma[1], \mathcal{B}[1])$ embedded in $\Sigma^1 \subset \Sigma$. The lengths are indicated. The green lines correspond to retraction lines of the corresponding product structures $\eta_1$ and $\eta_2$ and the orange lines correspond to retraction lines of the product structures chosen for the cylinders $\Sigma^2_{\Gamma[1]^1}$

Similarly to the construction of the graph $(\Gamma[1], \mathcal{B}[1])$, we construct $\Gamma[2] = \Gamma$. We observe that $\Sigma^2$ is an annulus whose boundary components are permuted by $\phi$. It is attached along an orbit of annuli $\mathcal{A}_{3,1}$ and $\mathcal{A}_{3,2}$ to $\Sigma^2$. It has total length equal to $4\pi/72 = \pi/18$ Since this orbit of annuli has screw number $-1/2$, we get that $\delta_2$ is the constant function $\pi/144$.

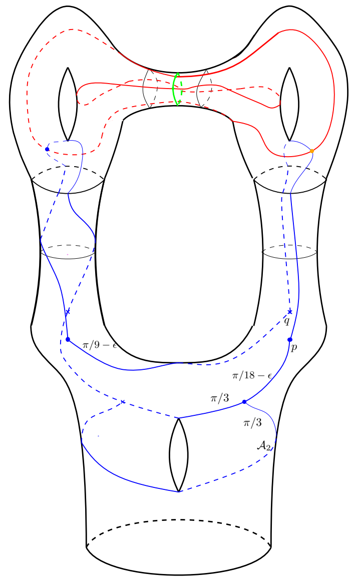


**Figure 6.7.** The final mixed tête-à-tête graph $(\Gamma^\bullet, \delta_\bullet)$. In green we have $\Gamma^2$; in red we have $\Gamma^1 \setminus \Gamma^2$ and in blue $\Gamma \setminus \Gamma^1$.

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