The geometric genus and Seiberg–Witten invariant of Newton nondegenerate surface singularities

PhD thesis

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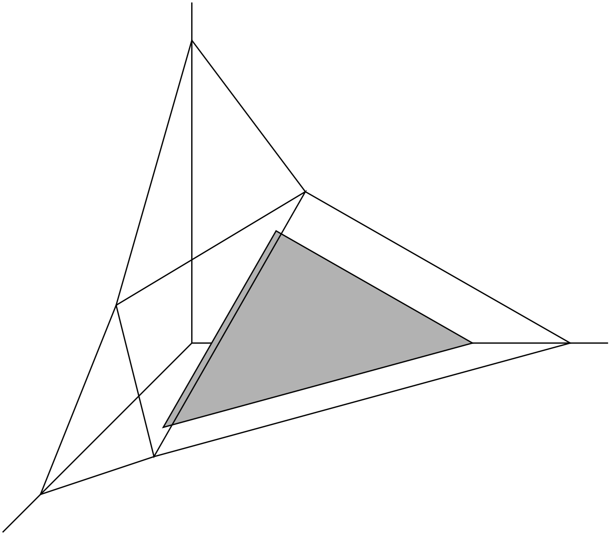

$$f(x) = x_1^4 + x_1^3 x_2^2 + x_2^{10} + x_1^2 x_3^3 + x_2^3 x_3^4 + x_3^8 = 0.$$

Computation sequences

Definition

Assume given a resolution graph G for a singularity $(X, 0)$. Let $Z \in L$ be an effective cycle. A *computation sequence* for Z is a sequence $Z_0, \ldots, Z_k$ so that $Z_0 = 0$, $Z_k = Z$ and for each i we have a $v(i) \in \mathcal{V}$ so that $Z_{i+1} = Z_i + E_{v(i)}$. Given such a computation sequence, its *continuation to infinity* is the sequence $(Z_i)_{i=0}^{\infty}$ recursively defined by $Z_{i+1} = Z_i + E_{v(i)}$ where we extend v to $\mathbb{N}$ by $v(i') = v(i)$ if $i' \equiv i (mod\, k)$.

Computation sequences

Theorem

Assume that each component E_v of the exceptional divisor is a rational curve and that $(X, 0)$ is Gorenstein. Let $Z \in L$ be an effective cycle and $(Z_i)_{i=0}^{k}$ a computation sequence for Z. Then

$$h_Z \leq \sum_{i=0}^{k-1} \max\{0, (-Z_i, E_{v(i)}) + 1\} \tag{1}$$

and we have an equality if and only if the natural maps $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_i)) \to H^0(E_{v(i)}, \mathcal{O}_{E_{v(i)}}(-Z_i))$ are surjective for all i.

Computation sequences and the geometric genus

In fact, let $H(t) = \sum_{l \in L} h_l t^l$ be the *Hilbert series* associated to the resolution $\tilde{X} \to X$, that is

$$h_l = \dim_{\mathbb{C}} \frac{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})}{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-l))}.$$

Then, the bound

$$\dim_{\mathbb{C}} \frac{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_i)}{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_{i+1}))} \leq \max\{0, (-Z_i, E_{v(i)}) + 1\}$$

follows from standard methods.

Theorem

We have

$$p_g = h_{Z_K}$$

where Z_K denotes the anticanonical cycle.

A computation sequence for Z_K

In the case of a Newton nondegenerate singularity, we construct a computation sequence $(Z_i)_{i=0}^k$ for the anticanonical cycle Z_K and a partition $(P_i)_{i=0}^k$ of the set of monomials x^p whose multiplicities along the exceptional divisor of $\tilde{X}$ are bounded by Z_K and prove

$$\dim_{\mathbb{C}} \frac{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_i))}{H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-Z_{i+1}))} = |P_i| = \max\{0, (-Z_i, E_{v(i)}) + 1\}.$$

The first equality is proved using a lemma of Ebeling and Gusein-Zade. For the second one, we prove a formula for the number of integral points in a dilated integral polygon in terms of its support functions. Then we show that the set P_i is of this form, and that the sum of support functions relates to the intersection number $(-Z_i, E_{v(i)})$.
This computation sequence can be explicitly computed from the minimal good resolution graph of $(X, 0)$, or, equivalently, the minimal plumbing graph for the link.

The Newton filtration

The *Newton Filtration* is a filtration of $\mathcal{O}_{\mathbb{C}^3,0}$ on one hand, and of $\mathcal{O}_{X,0} = \mathcal{O}_{\mathbb{C}^3,0}/(f)$ on the other hand. The associated Poincaré series satisfy

$$P^{\mathcal{A}}_{\mathbb{C}^3}(t) = \sum_{p\in\mathbb{N}^3} t^{\ell_f(p)}, \quad P^{\mathcal{A}}_X(t) = (1-t)P^{\mathcal{A}}_{\mathbb{C}^3}(t)$$

where ℓ_f is the concave function on $\mathbb{R}^3_{\geq 0}$ taking the value 1 on the Newton diagram, and restricting to a linear function on each ray from the origin. The nonpositive part of the spectrum $\mathrm{Sp}_{\leq 0}(f,0)$ and the geometric cenus can be calculated from $P_X(t)$.

An infinite computation sequence

A similar computation sequence $(Z_i)_{i=0}^{\infty}$ is constructed, in this case we obtain a partition $(P_i)_{i=0}^{\infty}$ of all monomials, or $\mathbb{N}^3$. For each i, we have a rational number r_i so that $\ell_f(p) = r_i$ for all $p \in P_i$.

Theorem

$$P_X^{\mathcal{A}}(t) = \sum_{i=0}^{\infty} \max\{0, (-Z_i, E_{v(i)}) + 1\} t^{r_i}.$$

This computation sequence can not be calculated directly from the link as the previous one. The necessary ingredient is the multiplicities of the function $x_1x_2x_3$ along the components of the exceptional divisor of the resolution $\tilde{X} \to X$.

The zeta function and the counting function

The *zeta function* is is a formal powerseries $Z_0(t) = \sum_{l \in L} z_l t^l$ defined in terms of a resolution graph of $(X, 0)$. It is therefore a *topological* invariant. We define the *counting function* $Q_0(t) = \sum_{l \in L} q_l t^l$ by setting

$$q_l = \sum \left\{ z_l \,\middle|\, l' \in L,\, l' \not\geq l \right\}.$$

Analytic and topological series

The zeta- and counting functions are motivated by the following facts. The *Poincaré series* associated with the resolution $\tilde{X} \to X$ is defined as $P(t) = -H(t)\prod_v (1 - t^{-1}) = \sum_{l \in L} p_l t^l$. The Hilbert series is then recovered from the Poincaré series by a formula analogous to the definition of the counting function: $h_l = \sum_{l' \not\geq l} p_l$ (Note that this is not a straight forward result, since there are elements in $\mathbb{Z}[[t^L]]$ killed by $1 - t_v^{-1}$).

Theorem (Némethi)

If $(X, 0)$ is a splice quotient singularity, then $Z_0(t) = P(t)$.

The Seiberg–Witten invariants

The Seiberg–Witten invariants are numbers $\mathbf{sw}_M(\sigma)$ associated to any $\mathrm{spin^c}$ structure σ on a three dimensional manifold. Being the link of a complex space, the link is equipped with the *canonical* $\mathrm{spin^c}$ *structure* σ_{can}. The above analogies between $H(t)$ and $Q(t)$, along with the following result, show a strong tie between the Seiberg–Witten invariant and the geometric genus.

Theorem

$$\mathbf{sw}_M(\sigma_{\mathrm{can}}) - \frac{Z_K + |\mathcal{V}|}{8} = q_{Z_K}.$$

The Seiberg–Witten invariant conjecture

The above formula for p_g was obtained by showing that for each i we have $h_{Z_{i+1}} - h_{Z_i} = \max\{0, (Z_i, E_{v(i)}) + 1\}$, and then taking sum over i. A similar approach works for the Seiberg–Witten invariant, proving the *Seiberg–Witten invariant conjecture* of Némethi and Nicolaescu in this case.

Theorem

Let $(Z_i)_{i=0}^k$ be the computation sequence for Z_K constructed above. Then, for each i

$$q_{Z_{i+1}} - q_{Z_i} = \max\{0, (Z_i, E_{v(i)}) + 1\}.$$

In particular,

$$\mathbf{sw}_M(\sigma_{\mathrm{can}}) - \frac{Z_K + |\mathcal{V}|}{8} = p_g.$$

Thank you!

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